

HUBBLE-NORMALISED ORTHONOMAL FRAME EQUATIONS FOR PERFECT FLUID COSMOLOGIES IN COMPONENT FORM

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1 General issues

1.1 Matter source

We consider a *perfect fluid* matter source with linear γ -law equation of state. The H -normalised matter variables relative to an H -normalised future-directed timelike reference congruence \mathfrak{d}_0 are Ω and

$$\begin{aligned}
 P &= \frac{1}{G_+} [(\gamma - 1) + (1 - \frac{2}{3}\gamma)(v_1^2 + v_2^2 + v_3^2)]\Omega & G_{\pm} &:= 1 \pm (\gamma - 1)(v_1^2 + v_2^2 + v_3^2) \\
 Q_1 &= \frac{\gamma}{G_+} \Omega v_1 & Q_2 &= \frac{\gamma}{G_+} \Omega v_2 & Q_3 &= \frac{\gamma}{G_+} \Omega v_3 \\
 \Pi_{11} &= \frac{\gamma}{3G_+} \Omega (2v_1^2 - v_2^2 - v_3^2) & \Pi_{22} &= \frac{\gamma}{3G_+} \Omega (2v_2^2 - v_3^2 - v_1^2) & \Pi_{33} &= \frac{\gamma}{3G_+} \Omega (2v_3^2 - v_1^2 - v_2^2) \\
 \Pi_{12} &= \frac{\gamma}{G_+} \Omega v_1 v_2 & \Pi_{23} &= \frac{\gamma}{G_+} \Omega v_2 v_3 & \Pi_{31} &= \frac{\gamma}{G_+} \Omega v_3 v_1
 \end{aligned}$$

1.2 Lagrangean and Eulerian viewpoints

The H -normalised frame derivatives are $\{\mathfrak{d}_a\}_{a=0,1,2,3} = \{\mathfrak{d}_0, \mathfrak{d}_\alpha\}_{\alpha=1,2,3}$. We can choose between two different viewpoints to express them relative to a local coordinate basis $\{\partial_\mu\}_{\mu=0,1,2,3} = \{\partial_t, \partial_i\}_{i=1,2,3}$.

- (i) The Lagrangean *fluid-comoving viewpoint*, which is specified by

$$v^\alpha = 0 \quad \Rightarrow \quad Q^\alpha = 0 \quad \Pi_{\alpha\beta} = 0,$$

defines

$$\mathfrak{d}_0 := \mathcal{M}^{-1} \partial_t \quad \mathfrak{d}_\alpha := E_\alpha^i (M_i \partial_t + \partial_i),$$

where \mathcal{M} denotes the H -normalised threading lapse function and M_i the coordinate components of the dimensionless threading shift 1-form field. Without loss of generality we can take $\partial_t M_i = 0$ (cf. Ref. [5]).

- (ii) The Eulerian $(3+1)$ -slicing viewpoint, which is specified by

$$W^\alpha = 0,$$

defines

$$\mathfrak{d}_0 := \mathcal{N}^{-1} (\partial_t - N^i \partial_i) \quad \mathfrak{d}_\alpha := E_\alpha^i \partial_i,$$

where \mathcal{N} denotes the H -normalised lapse function and N^i the coordinate components of the dimensionless shift vector field.

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2 Equation systems

2.1 Constraint equations

Commutator constraint equations:

$$0 = (\partial_2 - r_2 - A_2 - N_{31})\partial_3 - (\partial_3 - r_3 - A_3 + N_{12})\partial_2 - N_{11}\partial_1 - 2W_1\partial_0$$

$$0 = (\partial_3 - r_3 - A_3 - N_{12})\partial_1 - (\partial_1 - r_1 - A_1 + N_{23})\partial_3 - N_{22}\partial_2 - 2W_2\partial_0$$

$$0 = (\partial_1 - r_1 - A_1 - N_{23})\partial_2 - (\partial_2 - r_2 - A_2 + N_{31})\partial_1 - N_{33}\partial_3 - 2W_3\partial_0$$

Gauß constraint equation (iff $W^\alpha = 0$):

$$\begin{aligned} 0 = & 1 + \frac{1}{3}(2\partial_1 - 2r_1 - 3A_1)A_1 + \frac{1}{3}(2\partial_2 - 2r_2 - 3A_2)A_2 + \frac{1}{3}(2\partial_3 - 2r_3 - 3A_3)A_3 \\ & - \frac{1}{12}(N_{11}^2 + N_{22}^2 + N_{33}^2 - 2N_{11}N_{22} - 2N_{22}N_{33} - 2N_{33}N_{11}) - \frac{1}{3}(N_{12}^2 + N_{23}^2 + N_{31}^2) \\ & - \frac{1}{6}(\Sigma_{11}^2 + \Sigma_{22}^2 + \Sigma_{33}^2 + 2\Sigma_{12}^2 + 2\Sigma_{23}^2 + 2\Sigma_{31}^2) - \Omega - \Omega_\Lambda \\ & + \frac{1}{3}(W_1 - 2R_1)W_1 + \frac{1}{3}(W_2 - 2R_2)W_2 + \frac{1}{3}(W_3 - 2R_3)W_3 \end{aligned}$$

Codacci constraint equations (iff $W^\alpha = 0$):

$$\begin{aligned} 0 = & (\partial_1 - r_1)(2 - \Sigma_{11}) - (\partial_2 - r_2)(\Sigma_{12} + W_3) - (\partial_3 - r_3)(\Sigma_{31} - W_2) \\ & + 3A_1\Sigma_{11} + (3A_2 - N_{31})\Sigma_{12} + (3A_3 + N_{12})\Sigma_{31} + (N_{22} - N_{33})\Sigma_{23} - N_{23}(\Sigma_{22} - \Sigma_{33}) \\ & - 3Q_1 + N_{11}W_1 - (2\dot{U}_2 - A_2 - N_{31})W_3 + (2\dot{U}_3 - A_3 - N_{12})W_2 \end{aligned}$$

$$\begin{aligned} 0 = & (\partial_2 - r_2)(2 - \Sigma_{22}) - (\partial_3 - r_3)(\Sigma_{23} + W_1) - (\partial_1 - r_1)(\Sigma_{12} - W_3) \\ & + 3A_2\Sigma_{22} + (3A_3 - N_{12})\Sigma_{23} + (3A_1 + N_{23})\Sigma_{12} + (N_{33} - N_{11})\Sigma_{31} - N_{31}(\Sigma_{33} - \Sigma_{11}) \\ & - 3Q_2 + N_{22}W_2 - (2\dot{U}_3 - A_3 - N_{12})W_1 + (2\dot{U}_1 - A_1 - N_{23})W_3 \end{aligned}$$

$$\begin{aligned} 0 = & (\partial_3 - r_3)(2 - \Sigma_{33}) - (\partial_1 - r_1)(\Sigma_{31} + W_2) - (\partial_2 - r_2)(\Sigma_{23} - W_1) \\ & + 3A_3\Sigma_{33} + (3A_1 - N_{23})\Sigma_{31} + (3A_2 + N_{31})\Sigma_{23} + (N_{11} - N_{22})\Sigma_{12} - N_{12}(\Sigma_{11} - \Sigma_{22}) \\ & - 3Q_3 + N_{33}W_3 - (2\dot{U}_1 - A_1 - N_{23})W_2 + (2\dot{U}_2 - A_2 - N_{31})W_1 \end{aligned}$$

First Jacobi constraint equations:

$$\begin{aligned} 0 = & (\partial_1 - r_1 - 2A_1)N_{11} + (\partial_2 - r_2)(A_3 + N_{12}) - (\partial_3 - r_3)(A_2 - N_{31}) - 2(A_2N_{12} + A_3N_{31}) \\ & + 2(1 + \Sigma_{11})W_1 + 2(\Sigma_{31} + R_2)W_3 + 2(\Sigma_{12} - R_3)W_2 \end{aligned}$$

$$\begin{aligned} 0 = & (\partial_2 - r_2 - 2A_2)N_{22} + (\partial_3 - r_3)(A_1 + N_{23}) - (\partial_1 - r_1)(A_3 - N_{12}) - 2(A_3N_{23} + A_1N_{12}) \\ & + 2(1 + \Sigma_{22})W_2 + 2(\Sigma_{12} + R_3)W_1 + 2(\Sigma_{23} - R_1)W_3 \end{aligned}$$

$$\begin{aligned} 0 = & (\partial_3 - r_3 - 2A_3)N_{33} + (\partial_1 - r_1)(A_2 + N_{31}) - (\partial_2 - r_2)(A_1 - N_{23}) - 2(A_1N_{31} + A_2N_{23}) \\ & + 2(1 + \Sigma_{33})W_3 + 2(\Sigma_{23} + R_1)W_2 + 2(\Sigma_{31} - R_2)W_1 \end{aligned}$$

Second Jacobi constraint equation:

$$0 = (\partial_1 - r_1 - \dot{U}_1 - 2A_1)W_1 + (\partial_2 - r_2 - \dot{U}_2 - 2A_2)W_2 + (\partial_3 - r_3 - \dot{U}_3 - 2A_3)W_3$$

Constraint equations for Ω_Λ :

$$0 = (\partial_1 - 2r_1)\Omega_\Lambda$$

$$0 = (\partial_2 - 2r_2)\Omega_\Lambda$$

$$0 = (\partial_3 - 2r_3)\Omega_\Lambda$$

Remark: Note that for the (3+1)-slicing viewpoint (i.e., when the H -normalised future-directed timelike reference congruence ∂_0 is hypersurface orthogonal, $W^\alpha = 0$) the components of the frame gauge source functions \dot{U}^α and R^α do *not* show up in the constraint equations.

2.2 Evolution equations

Deceleration parameter:

$$q = \frac{1}{3}(\Sigma_{11}^2 + \Sigma_{22}^2 + \Sigma_{33}^2 + 2\Sigma_{12}^2 + 2\Sigma_{23}^2 + 2\Sigma_{31}^2) + \frac{1}{2}(\Omega + 3P) - \Omega_\Lambda - \frac{2}{3}(W_1^2 + W_2^2 + W_3^2) - \frac{1}{3}(\partial_1 - r_1 + \dot{U}_1 - 2A_1)\dot{U}_1 - \frac{1}{3}(\partial_2 - r_2 + \dot{U}_2 - 2A_2)\dot{U}_2 - \frac{1}{3}(\partial_3 - r_3 + \dot{U}_3 - 2A_3)\dot{U}_3$$

Evolution equations for E_α^i and gauge constraint equations:

$$\partial_0 \partial_1 = (q - \Sigma_{11})\partial_1 - (\Sigma_{31} - W_2 - R_2)\partial_3 - (\Sigma_{12} + W_3 + R_3)\partial_2 + \partial_1 \partial_0 - (r_1 - \dot{U}_1)\partial_0$$

$$\partial_0 \partial_2 = (q - \Sigma_{22})\partial_2 - (\Sigma_{12} - W_3 - R_3)\partial_1 - (\Sigma_{23} + W_1 + R_1)\partial_3 + \partial_2 \partial_0 - (r_2 - \dot{U}_2)\partial_0$$

$$\partial_0 \partial_3 = (q - \Sigma_{33})\partial_3 - (\Sigma_{23} - W_1 - R_1)\partial_2 - (\Sigma_{31} + W_2 + R_2)\partial_1 + \partial_3 \partial_0 - (r_3 - \dot{U}_3)\partial_0$$

Evolution equations for A^α :

$$\begin{aligned} \partial_0 A_1 &= (q - \Sigma_{11})A_1 - \frac{1}{2}(\partial_1 - r_1 + \dot{U}_1)(2 - \Sigma_{11}) \\ &\quad + \frac{1}{2}(\partial_2 - r_2 + \dot{U}_2 - 2A_2)(\Sigma_{12} + W_3 + R_3) + \frac{1}{2}(\partial_3 - r_3 + \dot{U}_3 - 2A_3)(\Sigma_{31} - W_2 - R_2) \end{aligned}$$

$$\begin{aligned} \partial_0 A_2 &= (q - \Sigma_{22})A_2 - \frac{1}{2}(\partial_2 - r_2 + \dot{U}_2)(2 - \Sigma_{22}) \\ &\quad + \frac{1}{2}(\partial_3 - r_3 + \dot{U}_3 - 2A_3)(\Sigma_{23} + W_1 + R_1) + \frac{1}{2}(\partial_1 - r_1 + \dot{U}_1 - 2A_1)(\Sigma_{12} - W_3 - R_3) \end{aligned}$$

$$\begin{aligned} \partial_0 A_3 &= (q - \Sigma_{33})A_3 - \frac{1}{2}(\partial_3 - r_3 + \dot{U}_3)(2 - \Sigma_{33}) \\ &\quad + \frac{1}{2}(\partial_1 - r_1 + \dot{U}_1 - 2A_1)(\Sigma_{31} + W_2 + R_2) + \frac{1}{2}(\partial_2 - r_2 + \dot{U}_2 - 2A_2)(\Sigma_{23} - W_1 - R_1) \end{aligned}$$

Evolution equations for r_α :

$$\partial_0 r_1 = (q - \Sigma_{11})r_1 - (\Sigma_{31} - W_2 - R_2)r_3 - (\Sigma_{12} + W_3 + R_3)r_2 + (\partial_1 - r_1 + \dot{U}_1)(q + 1)$$

$$\partial_0 r_2 = (q - \Sigma_{22})r_2 - (\Sigma_{12} - W_3 - R_3)r_1 - (\Sigma_{23} + W_1 + R_1)r_3 + (\partial_2 - r_2 + \dot{U}_2)(q + 1)$$

$$\partial_0 r_3 = (q - \Sigma_{33})r_3 - (\Sigma_{23} - W_1 - R_1)r_2 - (\Sigma_{31} + W_2 + R_2)r_1 + (\partial_3 - r_3 + \dot{U}_3)(q + 1)$$

Evolution equations for W^α :

$$\begin{aligned} \partial_0 W_1 &= (q - 1 + \Sigma_{11})W_1 + (\Sigma_{31} + R_2)W_3 + (\Sigma_{12} - R_3)W_2 \\ &\quad - \frac{1}{2}N_{11}\dot{U}_1 + \frac{1}{2}(\partial_2 - r_2 - A_2 - N_{31})\dot{U}_3 - \frac{1}{2}(\partial_3 - r_3 - A_3 + N_{12})\dot{U}_2 \end{aligned}$$

$$\begin{aligned} \partial_0 W_2 &= (q - 1 + \Sigma_{22})W_2 + (\Sigma_{12} + R_3)W_1 + (\Sigma_{23} - R_1)W_3 \\ &\quad - \frac{1}{2}N_{22}\dot{U}_2 + \frac{1}{2}(\partial_3 - r_3 - A_3 - N_{12})\dot{U}_1 - \frac{1}{2}(\partial_1 - r_1 - A_1 + N_{23})\dot{U}_3 \end{aligned}$$

$$\begin{aligned} \partial_0 W_3 &= (q - 1 + \Sigma_{33})W_3 + (\Sigma_{23} + R_1)W_2 + (\Sigma_{31} - R_2)W_1 \\ &\quad - \frac{1}{2}N_{33}\dot{U}_3 + \frac{1}{2}(\partial_1 - r_1 - A_1 - N_{23})\dot{U}_2 - \frac{1}{2}(\partial_2 - r_2 - A_2 + N_{31})\dot{U}_1 \end{aligned}$$

Evolution equations for $\Sigma_{\alpha\beta}$:

$$\begin{aligned}\partial_0 \Sigma_{11} &= (q-2)\Sigma_{11} - \frac{1}{3}(2N_{11}^2 - N_{22}^2 - N_{33}^2 + 2N_{22}N_{33} - N_{33}N_{11} - N_{11}N_{22}) \\ &+ \frac{2}{3}(2N_{23}^2 - N_{31}^2 - N_{12}^2) - \frac{1}{3}[2(\partial_1 - r_1)A_1 - (\partial_2 - r_2)A_2 - (\partial_3 - r_3)A_3] \\ &+ (\partial_2 - r_2 + \dot{U}_2 - 2A_2)N_{31} - (\partial_3 - r_3 + \dot{U}_3 - 2A_3)N_{12} + 2(R_2\Sigma_{31} - R_3\Sigma_{12}) + 3\Pi_{11} \\ &+ \frac{1}{3}[2(\partial_1 - r_1 + \dot{U}_1 + A_1)\dot{U}_1 - (\partial_2 - r_2 + \dot{U}_2 + A_2)\dot{U}_2 - (\partial_3 - r_3 + \dot{U}_3 + A_3)\dot{U}_3] \\ &- \frac{2}{3}(2R_1W_1 - R_2W_2 - R_3W_3)\end{aligned}$$

$$\begin{aligned}\partial_0 \Sigma_{22} &= (q-2)\Sigma_{22} - \frac{1}{3}(2N_{22}^2 - N_{33}^2 - N_{11}^2 + 2N_{33}N_{11} - N_{11}N_{22} - N_{22}N_{33}) \\ &+ \frac{2}{3}(2N_{31}^2 - N_{12}^2 - N_{23}^2) - \frac{1}{3}[2(\partial_2 - r_2)A_2 - (\partial_3 - r_3)A_3 - (\partial_1 - r_1)A_1] \\ &+ (\partial_3 - r_3 + \dot{U}_3 - 2A_3)N_{12} - (\partial_1 - r_1 + \dot{U}_1 - 2A_1)N_{23} + 2(R_3\Sigma_{12} - R_1\Sigma_{23}) + 3\Pi_{22} \\ &+ \frac{1}{3}[2(\partial_2 - r_2 + \dot{U}_2 + A_2)\dot{U}_2 - (\partial_3 - r_3 + \dot{U}_3 + A_3)\dot{U}_3 - (\partial_1 - r_1 + \dot{U}_1 + A_1)\dot{U}_1] \\ &- \frac{2}{3}(2R_2W_2 - R_3W_3 - R_1W_1)\end{aligned}$$

$$\begin{aligned}\partial_0 \Sigma_{33} &= (q-2)\Sigma_{33} - \frac{1}{3}(2N_{33}^2 - N_{11}^2 - N_{22}^2 + 2N_{11}N_{22} - N_{22}N_{33} - N_{33}N_{11}) \\ &+ \frac{2}{3}(2N_{12}^2 - N_{23}^2 - N_{31}^2) - \frac{1}{3}[2(\partial_3 - r_3)A_3 - (\partial_1 - r_1)A_1 - (\partial_2 - r_2)A_2] \\ &+ (\partial_1 - r_1 + \dot{U}_1 - 2A_1)N_{23} - (\partial_2 - r_2 + \dot{U}_2 - 2A_2)N_{31} + 2(R_1\Sigma_{23} - R_2\Sigma_{31}) + 3\Pi_{33} \\ &+ \frac{1}{3}[2(\partial_3 - r_3 + \dot{U}_3 + A_3)\dot{U}_3 - (\partial_1 - r_1 + \dot{U}_1 + A_1)\dot{U}_1 - (\partial_2 - r_2 + \dot{U}_2 + A_2)\dot{U}_2] \\ &- \frac{2}{3}(2R_3W_3 - R_1W_1 - R_2W_2)\end{aligned}$$

$$\begin{aligned}\partial_0 \Sigma_{12} &= (q-2)\Sigma_{12} + (N_{33} - N_{11} - N_{22})N_{12} - 2N_{23}N_{31} - \frac{1}{2}(\partial_1 - r_1)A_2 - \frac{1}{2}(\partial_2 - r_2)A_1 \\ &+ \frac{1}{2}(\partial_3 - r_3 + \dot{U}_3 - 2A_3)(N_{11} - N_{22}) - \frac{1}{2}(\partial_1 - r_1 + \dot{U}_1 - 2A_1)N_{31} \\ &+ \frac{1}{2}(\partial_2 - r_2 + \dot{U}_2 - 2A_2)N_{23} + R_3(\Sigma_{11} - \Sigma_{22}) - R_1\Sigma_{31} + R_2\Sigma_{23} + 3\Pi_{12} \\ &+ \frac{1}{2}(\partial_1 - r_1 + \dot{U}_1 + A_1)\dot{U}_2 + \frac{1}{2}(\partial_2 - r_2 + \dot{U}_2 + A_2)\dot{U}_1 - (R_1W_2 + R_2W_1)\end{aligned}$$

$$\begin{aligned}\partial_0 \Sigma_{23} &= (q-2)\Sigma_{23} + (N_{11} - N_{22} - N_{33})N_{23} - 2N_{31}N_{12} - \frac{1}{2}(\partial_2 - r_2)A_3 - \frac{1}{2}(\partial_3 - r_3)A_2 \\ &+ \frac{1}{2}(\partial_1 - r_1 + \dot{U}_1 - 2A_1)(N_{22} - N_{33}) - \frac{1}{2}(\partial_2 - r_2 + \dot{U}_2 - 2A_2)N_{12} \\ &+ \frac{1}{2}(\partial_3 - r_3 + \dot{U}_3 - 2A_3)N_{31} + R_1(\Sigma_{22} - \Sigma_{33}) - R_2\Sigma_{12} + R_3\Sigma_{31} + 3\Pi_{23} \\ &+ \frac{1}{2}(\partial_2 - r_2 + \dot{U}_2 + A_2)\dot{U}_3 + \frac{1}{2}(\partial_3 - r_3 + \dot{U}_3 + A_3)\dot{U}_2 - (R_2W_3 + R_3W_2)\end{aligned}$$

$$\begin{aligned}\partial_0 \Sigma_{31} &= (q-2)\Sigma_{31} + (N_{22} - N_{33} - N_{11})N_{31} - 2N_{12}N_{23} - \frac{1}{2}(\partial_3 - r_3)A_1 - \frac{1}{2}(\partial_1 - r_1)A_3 \\ &+ \frac{1}{2}(\partial_2 - r_2 + \dot{U}_2 - 2A_2)(N_{33} - N_{11}) - \frac{1}{2}(\partial_3 - r_3 + \dot{U}_3 - 2A_3)N_{23} \\ &+ \frac{1}{2}(\partial_1 - r_1 + \dot{U}_1 - 2A_1)N_{12} + R_2(\Sigma_{33} - \Sigma_{11}) - R_3\Sigma_{23} + R_1\Sigma_{12} + 3\Pi_{31} \\ &+ \frac{1}{2}(\partial_3 - r_3 + \dot{U}_3 + A_3)\dot{U}_1 + \frac{1}{2}(\partial_1 - r_1 + \dot{U}_1 + A_1)\dot{U}_3 - (R_3W_1 + R_1W_3)\end{aligned}$$

∂_3 -adapted (1+2)-decomposition of $\Sigma_{\alpha\beta}$:

$$\begin{aligned}\Sigma_+ &:= \frac{1}{2}(\Sigma_{11} + \Sigma_{22}) & \Sigma_{11} &= \Sigma_+ + \sqrt{3}\Sigma_- \\ \Sigma_- &:= \frac{1}{2\sqrt{3}}(\Sigma_{11} - \Sigma_{22}) & \Sigma_{22} &= \Sigma_+ - \sqrt{3}\Sigma_- \\ \Sigma_\times &:= \frac{1}{\sqrt{3}}\Sigma_{12} & \Sigma_{33} &= -2\Sigma_+ \\ \Sigma_1 &:= \frac{1}{\sqrt{3}}\Sigma_{23} & \Sigma_{12} &= \sqrt{3}\Sigma_\times \\ \Sigma_2 &:= \frac{1}{\sqrt{3}}\Sigma_{31} & \Sigma_{23} &= \sqrt{3}\Sigma_1 \\ & & \Sigma_{31} &= \sqrt{3}\Sigma_2\end{aligned}$$

Squared scalar of $\Sigma_{\alpha\beta}$:

$$\Sigma^2 = \frac{1}{6}(\Sigma_{11}^2 + \Sigma_{22}^2 + \Sigma_{33}^2 + 2\Sigma_{12}^2 + 2\Sigma_{23}^2 + 2\Sigma_{31}^2) = \Sigma_+^2 + \Sigma_-^2 + \Sigma_\times^2 + \Sigma_1^2 + \Sigma_2^2$$

Evolution equations for $N_{\alpha\beta}$:

$$\begin{aligned}\partial_0 N_{11} &= (q + 2\Sigma_{11})N_{11} \\ &\quad - (\partial_2 - r_2 + \dot{U}_2 - 2N_{31})(\Sigma_{31} + W_2 + R_2) + (\partial_3 - r_3 + \dot{U}_3 + 2N_{12})(\Sigma_{12} - W_3 - R_3)\end{aligned}$$

$$\begin{aligned}\partial_0 N_{22} &= (q + 2\Sigma_{22})N_{22} \\ &\quad - (\partial_3 - r_3 + \dot{U}_3 - 2N_{12})(\Sigma_{12} + W_3 + R_3) + (\partial_1 - r_1 + \dot{U}_1 + 2N_{23})(\Sigma_{23} - W_1 - R_1)\end{aligned}$$

$$\begin{aligned}\partial_0 N_{33} &= (q + 2\Sigma_{33})N_{33} \\ &\quad - (\partial_1 - r_1 + \dot{U}_1 - 2N_{23})(\Sigma_{23} + W_1 + R_1) + (\partial_2 - r_2 + \dot{U}_2 + 2N_{31})(\Sigma_{31} - W_2 - R_2)\end{aligned}$$

$$\begin{aligned}\partial_0 N_{12} &= (q - \Sigma_{33})N_{12} + \Sigma_{12}(N_{11} + N_{22}) + (W_3 + R_3)(N_{11} - N_{22}) \\ &\quad - \frac{1}{2}(\partial_3 - r_3 + \dot{U}_3)(\Sigma_{11} - \Sigma_{22}) + \frac{1}{2}(\partial_1 - r_1 + \dot{U}_1 + 2N_{23})(\Sigma_{31} + W_2 + R_2) \\ &\quad - \frac{1}{2}(\partial_2 - r_2 + \dot{U}_2 - 2N_{31})(\Sigma_{23} - W_1 - R_1)\end{aligned}$$

$$\begin{aligned}\partial_0 N_{23} &= (q - \Sigma_{11})N_{23} + \Sigma_{23}(N_{22} + N_{33}) + (W_1 + R_1)(N_{22} - N_{33}) \\ &\quad - \frac{1}{2}(\partial_1 - r_1 + \dot{U}_1)(\Sigma_{22} - \Sigma_{33}) + \frac{1}{2}(\partial_2 - r_2 + \dot{U}_2 + 2N_{31})(\Sigma_{12} + W_3 + R_3) \\ &\quad - \frac{1}{2}(\partial_3 - r_3 + \dot{U}_3 - 2N_{12})(\Sigma_{31} - W_2 - R_2)\end{aligned}$$

$$\begin{aligned}\partial_0 N_{31} &= (q - \Sigma_{22})N_{31} + \Sigma_{31}(N_{33} + N_{11}) + (W_2 + R_2)(N_{33} - N_{11}) \\ &\quad - \frac{1}{2}(\partial_2 - r_2 + \dot{U}_2)(\Sigma_{33} - \Sigma_{11}) + \frac{1}{2}(\partial_3 - r_3 + \dot{U}_3 + 2N_{12})(\Sigma_{23} + W_1 + R_1) \\ &\quad - \frac{1}{2}(\partial_1 - r_1 + \dot{U}_1 - 2N_{23})(\Sigma_{12} - W_3 - R_3)\end{aligned}$$

Evolution equation for Ω :

$$\begin{aligned}\partial_0 \Omega &= (2q - 1)\Omega - 3P - (\partial_1 - 2r_1 + 2\dot{U}_1 - 2A_1)Q_1 - (\partial_2 - 2r_2 + 2\dot{U}_2 - 2A_2)Q_2 \\ &\quad - (\partial_3 - 2r_3 + 2\dot{U}_3 - 2A_3)Q_3 \\ &\quad - (\Sigma_{11}\Pi_{11} + \Sigma_{22}\Pi_{22} + \Sigma_{33}\Pi_{33} + 2\Sigma_{12}\Pi_{12} + 2\Sigma_{23}\Pi_{23} + 2\Sigma_{31}\Pi_{31})\end{aligned}$$

Evolution equations for Q^α :

$$\begin{aligned}\partial_0 Q_1 &= (2q - 2 - \Sigma_{11})Q_1 - (\Sigma_{31} + W_2 - R_2)Q_3 - (\Sigma_{12} - W_3 + R_3)Q_2 - (\Omega + P)\dot{U}_1 \\ &\quad + (N_{22} - N_{33})\Pi_{23} - N_{23}(\Pi_{22} - \Pi_{33}) - (\partial_1 - 2r_1)P - (\partial_1 - 2r_1 + \dot{U}_1 - 3A_1)\Pi_{11} \\ &\quad - (\partial_2 - 2r_2 + \dot{U}_2 - 3A_2 + N_{31})\Pi_{12} - (\partial_3 - 2r_3 + \dot{U}_3 - 3A_3 - N_{12})\Pi_{31}\end{aligned}$$

$$\begin{aligned}\partial_0 Q_2 &= (2q - 2 - \Sigma_{22})Q_2 - (\Sigma_{12} + W_3 - R_3)Q_1 - (\Sigma_{23} - W_1 + R_1)Q_3 - (\Omega + P)\dot{U}_2 \\ &\quad + (N_{33} - N_{11})\Pi_{31} - N_{31}(\Pi_{33} - \Pi_{11}) - (\partial_2 - 2r_2)P - (\partial_2 - 2r_2 + \dot{U}_2 - 3A_2)\Pi_{22} \\ &\quad - (\partial_3 - 2r_3 + \dot{U}_3 - 3A_3 + N_{12})\Pi_{23} - (\partial_1 - 2r_1 + \dot{U}_1 - 3A_1 - N_{23})\Pi_{12}\end{aligned}$$

$$\begin{aligned}\partial_0 Q_3 &= (2q - 2 - \Sigma_{33})Q_3 - (\Sigma_{23} + W_1 - R_1)Q_2 - (\Sigma_{31} - W_2 + R_2)Q_1 - (\Omega + P)\dot{U}_3 \\ &\quad + (N_{11} - N_{22})\Pi_{12} - N_{12}(\Pi_{11} - \Pi_{22}) - (\partial_3 - 2r_3)P - (\partial_3 - 2r_3 + \dot{U}_3 - 3A_3)\Pi_{33} \\ &\quad - (\partial_1 - 2r_1 + \dot{U}_1 - 3A_1 + N_{23})\Pi_{31} - (\partial_2 - 2r_2 + \dot{U}_2 - 3A_2 - N_{31})\Pi_{23}\end{aligned}$$

Evolution equation for Ω_Λ :

$$\partial_0 \Omega_\Lambda = 2(q + 1)\Omega_\Lambda$$

3 Weyl curvature

Components of $\mathcal{E}_{\alpha\beta}$:

$$\begin{aligned}\mathcal{E}_{11} = & -\frac{1}{3}(\mathfrak{d}_2 - r_2 - 2A_2)N_{31} + \frac{1}{3}(\mathfrak{d}_3 - r_3 - 2A_3)N_{12} \\ & + \frac{1}{9}[2(\mathfrak{d}_1 - r_1)A_1 - (\mathfrak{d}_2 - r_2)A_2 - (\mathfrak{d}_3 - r_3)A_3] \\ & + \frac{1}{9}(2N_{11}^2 - N_{22}^2 - N_{33}^2 + 2N_{22}N_{33} - N_{33}N_{11} - N_{11}N_{22}) - \frac{2}{9}(2N_{23}^2 - N_{31}^2 - N_{12}^2) \\ & + \frac{1}{3}\Sigma_{11} - \frac{1}{9}(2\Sigma_{11}^2 - \Sigma_{22}^2 - \Sigma_{33}^2) + \frac{1}{9}(2\Sigma_{23}^2 - \Sigma_{31}^2 - \Sigma_{12}^2) - \frac{\Upsilon}{6G_+}\Omega(2v_1^2 - v_2^2 - v_3^2) \\ & - \frac{2}{9}(W_1 - 2R_1)W_1 + \frac{1}{9}(W_2 - 2R_2)W_2 + \frac{1}{9}(W_3 - 2R_3)W_3\end{aligned}$$

$$\begin{aligned}\mathcal{E}_{22} = & -\frac{1}{3}(\mathfrak{d}_3 - r_3 - 2A_3)N_{12} + \frac{1}{3}(\mathfrak{d}_1 - r_1 - 2A_1)N_{23} \\ & + \frac{1}{9}[2(\mathfrak{d}_2 - r_2)A_2 - (\mathfrak{d}_3 - r_3)A_3 - (\mathfrak{d}_1 - r_1)A_1] \\ & + \frac{1}{9}(2N_{22}^2 - N_{33}^2 - N_{11}^2 + 2N_{33}N_{11} - N_{11}N_{22} - N_{22}N_{33}) - \frac{2}{9}(2N_{31}^2 - N_{12}^2 - N_{23}^2) \\ & + \frac{1}{3}\Sigma_{22} - \frac{1}{9}(2\Sigma_{22}^2 - \Sigma_{33}^2 - \Sigma_{11}^2) + \frac{1}{9}(2\Sigma_{31}^2 - \Sigma_{12}^2 - \Sigma_{23}^2) - \frac{\Upsilon}{6G_+}\Omega(2v_2^2 - v_3^2 - v_1^2) \\ & - \frac{2}{9}(W_2 - 2R_2)W_2 + \frac{1}{9}(W_3 - 2R_3)W_3 + \frac{1}{9}(W_1 - 2R_1)W_1\end{aligned}$$

$$\begin{aligned}\mathcal{E}_{33} = & -\frac{1}{3}(\mathfrak{d}_1 - r_1 - 2A_1)N_{23} + \frac{1}{3}(\mathfrak{d}_2 - r_2 - 2A_2)N_{31} \\ & + \frac{1}{9}[2(\mathfrak{d}_3 - r_3)A_3 - (\mathfrak{d}_1 - r_1)A_1 - (\mathfrak{d}_2 - r_2)A_2] \\ & + \frac{1}{9}(2N_{33}^2 - N_{11}^2 - N_{22}^2 + 2N_{11}N_{22} - N_{22}N_{33} - N_{33}N_{11}) - \frac{2}{9}(2N_{12}^2 - N_{23}^2 - N_{31}^2) \\ & + \frac{1}{3}\Sigma_{33} - \frac{1}{9}(2\Sigma_{33}^2 - \Sigma_{11}^2 - \Sigma_{22}^2) + \frac{1}{9}(2\Sigma_{12}^2 - \Sigma_{23}^2 - \Sigma_{31}^2) - \frac{\Upsilon}{6G_+}\Omega(2v_3^2 - v_1^2 - v_2^2) \\ & - \frac{2}{9}(W_3 - 2R_3)W_3 + \frac{1}{9}(W_1 - 2R_1)W_1 + \frac{1}{9}(W_2 - 2R_2)W_2\end{aligned}$$

$$\begin{aligned}\mathcal{E}_{12} = & -\frac{1}{6}(\mathfrak{d}_3 - r_3 - 2A_3)(N_{11} - N_{22}) + \frac{1}{6}(\mathfrak{d}_1 - r_1 - 2A_1)N_{31} - \frac{1}{6}(\mathfrak{d}_2 - r_2 - 2A_2)N_{23} \\ & + \frac{1}{6}(\mathfrak{d}_1 - r_1)A_2 + \frac{1}{6}(\mathfrak{d}_2 - r_2)A_1 - \frac{1}{3}(N_{33} - N_{11} - N_{22})N_{12} + \frac{2}{3}N_{23}N_{31} \\ & + \frac{1}{3}(1 - \Sigma_{11} - \Sigma_{22})\Sigma_{12} - \frac{1}{3}\Sigma_{23}\Sigma_{31} - \frac{\Upsilon}{2G_+}\Omega v_1 v_2 \\ & - \frac{1}{6}(W_1 - 2R_1)W_2 - \frac{1}{6}(W_2 - 2R_2)W_1\end{aligned}$$

$$\begin{aligned}\mathcal{E}_{23} = & -\frac{1}{6}(\mathfrak{d}_1 - r_1 - 2A_1)(N_{22} - N_{33}) + \frac{1}{6}(\mathfrak{d}_2 - r_2 - 2A_2)N_{12} - \frac{1}{6}(\mathfrak{d}_3 - r_3 - 2A_3)N_{31} \\ & + \frac{1}{6}(\mathfrak{d}_2 - r_2)A_3 + \frac{1}{6}(\mathfrak{d}_3 - r_3)A_2 - \frac{1}{3}(N_{11} - N_{22} - N_{33})N_{23} + \frac{2}{3}N_{31}N_{12} \\ & + \frac{1}{3}(1 - \Sigma_{22} - \Sigma_{33})\Sigma_{23} - \frac{1}{3}\Sigma_{31}\Sigma_{12} - \frac{\Upsilon}{2G_+}\Omega v_2 v_3 \\ & - \frac{1}{6}(W_2 - 2R_2)W_3 - \frac{1}{6}(W_3 - 2R_3)W_2\end{aligned}$$

$$\begin{aligned}\mathcal{E}_{31} = & -\frac{1}{6}(\mathfrak{d}_2 - r_2 - 2A_2)(N_{33} - N_{11}) + \frac{1}{6}(\mathfrak{d}_3 - r_3 - 2A_3)N_{23} - \frac{1}{6}(\mathfrak{d}_1 - r_1 - 2A_1)N_{12} \\ & + \frac{1}{6}(\mathfrak{d}_3 - r_3)A_1 + \frac{1}{6}(\mathfrak{d}_1 - r_1)A_3 - \frac{1}{3}(N_{22} - N_{33} - N_{11})N_{31} + \frac{2}{3}N_{12}N_{23} \\ & + \frac{1}{3}(1 - \Sigma_{33} - \Sigma_{11})\Sigma_{31} - \frac{1}{3}\Sigma_{12}\Sigma_{23} - \frac{\Upsilon}{2G_+}\Omega v_3 v_1 \\ & - \frac{1}{6}(W_3 - 2R_3)W_1 - \frac{1}{6}(W_1 - 2R_1)W_3\end{aligned}$$

Remark: A \mathfrak{d}_3 -adapted (1+2)-decomposition of $\mathcal{E}_{\alpha\beta}$ equivalent to that for $\Sigma_{\alpha\beta}$ can be employed.

Components of $\mathcal{H}_{\alpha\beta}$:

$$\begin{aligned}\mathcal{H}_{11} &= \frac{1}{3}(\mathfrak{d}_2 - r_2 - A_2)\Sigma_{31} - \frac{1}{3}(\mathfrak{d}_3 - r_3 - A_3)\Sigma_{12} - \frac{1}{2}(N_{11} - N_{22} - N_{33})\Sigma_{11} \\ &\quad - \frac{1}{3}N_{22}(\Sigma_{11} - \Sigma_{22}) + \frac{1}{3}N_{33}(\Sigma_{33} - \Sigma_{11}) + \frac{1}{3}(2N_{23}\Sigma_{23} - N_{31}\Sigma_{31} - N_{12}\Sigma_{12}) \\ &\quad - \frac{2}{9}(\mathfrak{d}_1 - r_1 + 2\dot{U}_1 + A_1)W_1 \\ &\quad + \frac{1}{9}(\mathfrak{d}_2 - r_2 + 2\dot{U}_2 + A_2 - 3N_{31})W_2 + \frac{1}{9}(\mathfrak{d}_3 - r_3 + 2\dot{U}_3 + A_3 - 3N_{12})W_3\end{aligned}$$

$$\begin{aligned}\mathcal{H}_{22} &= \frac{1}{3}(\mathfrak{d}_3 - r_3 - A_3)\Sigma_{12} - \frac{1}{3}(\mathfrak{d}_1 - r_1 - A_1)\Sigma_{23} - \frac{1}{2}(N_{22} - N_{33} - N_{11})\Sigma_{22} \\ &\quad - \frac{1}{3}N_{33}(\Sigma_{22} - \Sigma_{33}) + \frac{1}{3}N_{11}(\Sigma_{11} - \Sigma_{22}) + \frac{1}{3}(2N_{31}\Sigma_{31} - N_{12}\Sigma_{12} - N_{23}\Sigma_{23}) \\ &\quad - \frac{2}{9}(\mathfrak{d}_2 - r_2 + 2\dot{U}_2 + A_2)W_2 \\ &\quad + \frac{1}{9}(\mathfrak{d}_3 - r_3 + 2\dot{U}_3 + A_3 - 3N_{12})W_3 + \frac{1}{9}(\mathfrak{d}_1 - r_1 + 2\dot{U}_1 + A_1 - 3N_{23})W_1\end{aligned}$$

$$\begin{aligned}\mathcal{H}_{33} &= \frac{1}{3}(\mathfrak{d}_1 - r_1 - A_1)\Sigma_{23} - \frac{1}{3}(\mathfrak{d}_2 - r_2 - A_2)\Sigma_{31} - \frac{1}{2}(N_{33} - N_{11} - N_{22})\Sigma_{33} \\ &\quad - \frac{1}{3}N_{11}(\Sigma_{33} - \Sigma_{11}) + \frac{1}{3}N_{22}(\Sigma_{22} - \Sigma_{33}) + \frac{1}{3}(2N_{12}\Sigma_{12} - N_{23}\Sigma_{23} - N_{31}\Sigma_{31}) \\ &\quad - \frac{2}{9}(\mathfrak{d}_3 - r_3 + 2\dot{U}_3 + A_3)W_3 \\ &\quad + \frac{1}{9}(\mathfrak{d}_1 - r_1 + 2\dot{U}_1 + A_1 - 3N_{23})W_1 + \frac{1}{9}(\mathfrak{d}_2 - r_2 + 2\dot{U}_2 + A_2 - 3N_{31})W_2\end{aligned}$$

$$\begin{aligned}\mathcal{H}_{12} &= \frac{1}{6}(\mathfrak{d}_3 - r_3 - A_3)(\Sigma_{11} - \Sigma_{22}) - \frac{1}{6}(\mathfrak{d}_1 - r_1 - A_1 + 3N_{23})\Sigma_{31} + \frac{1}{6}(\mathfrak{d}_2 - r_2 - A_2 - 3N_{31})\Sigma_{23} \\ &\quad + \frac{1}{6}(N_{33} - 2N_{11} - 2N_{22})\Sigma_{12} - \frac{1}{2}N_{12}(\Sigma_{11} + \Sigma_{22}) \\ &\quad - \frac{1}{6}(\mathfrak{d}_1 - r_1 + 2\dot{U}_1 + A_1 + N_{23})W_2 - \frac{1}{6}(\mathfrak{d}_2 - r_2 + 2\dot{U}_2 + A_2 - N_{31})W_1 - \frac{1}{6}(N_{11} - N_{22})W_3\end{aligned}$$

$$\begin{aligned}\mathcal{H}_{23} &= \frac{1}{6}(\mathfrak{d}_1 - r_1 - A_1)(\Sigma_{22} - \Sigma_{33}) - \frac{1}{6}(\mathfrak{d}_2 - r_2 - A_2 + 3N_{31})\Sigma_{12} + \frac{1}{6}(\mathfrak{d}_3 - r_3 - A_3 - 3N_{12})\Sigma_{31} \\ &\quad + \frac{1}{6}(N_{11} - 2N_{22} - 2N_{33})\Sigma_{23} - \frac{1}{2}N_{23}(\Sigma_{22} + \Sigma_{33}) \\ &\quad - \frac{1}{6}(\mathfrak{d}_2 - r_2 + 2\dot{U}_2 + A_2 + N_{31})W_3 - \frac{1}{6}(\mathfrak{d}_3 - r_3 + 2\dot{U}_3 + A_3 - N_{12})W_2 - \frac{1}{6}(N_{22} - N_{33})W_1\end{aligned}$$

$$\begin{aligned}\mathcal{H}_{31} &= \frac{1}{6}(\mathfrak{d}_2 - r_2 - A_2)(\Sigma_{33} - \Sigma_{11}) - \frac{1}{6}(\mathfrak{d}_3 - r_3 - A_3 + 3N_{12})\Sigma_{23} + \frac{1}{6}(\mathfrak{d}_1 - r_1 - A_1 - 3N_{23})\Sigma_{12} \\ &\quad + \frac{1}{6}(N_{22} - 2N_{33} - 2N_{11})\Sigma_{31} - \frac{1}{2}N_{31}(\Sigma_{33} + \Sigma_{11}) \\ &\quad - \frac{1}{6}(\mathfrak{d}_3 - r_3 + 2\dot{U}_3 + A_3 + N_{12})W_1 - \frac{1}{6}(\mathfrak{d}_1 - r_1 + 2\dot{U}_1 + A_1 - N_{23})W_3 - \frac{1}{6}(N_{33} - N_{11})W_2\end{aligned}$$

Remark: A \mathfrak{d}_3 -adapted (1+2)-decomposition of $\mathcal{H}_{\alpha\beta}$ equivalent to that for $\Sigma_{\alpha\beta}$ can be employed.

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